Gaussian elimination

In the last example last week, we started with a system of equations, converted it into an augmented matrix, applied elementary operations one by one, until the result was a "nice" matrix, of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right].$$

If the number of equations is < the # of variables, we get a slightly different looking "nice" matrix.

Ex: $2x + y + z = 5$
$2x - y + 7z = 0$ $\rightsquigarrow$ $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 2 & -1 & 7 & 0 \end{array}\right]$

$$\xrightarrow{② - ①} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 0 & -2 & 6 & -4 \end{array}\right] \xrightarrow{\frac{1}{2}①} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 2 \\ 0 & -2 & 6 & -4 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}②} \underbrace{\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 2 \\ 0 & 1 & -3 & 2 \end{array}\right]}_{\text{row-echelon}} \xrightarrow{① - \frac{1}{2}②} \left.\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 \end{array}\right]\right\} \text{reduced row-echelon}$$

rows begin w/ ones

0 above leading one

$\rightsquigarrow$ $x + z = 1$
$y - 3z = 2$

Now we can easily solve for x and y in terms of z:

$x = 1 - z$, $y = 2 + 3z$.

Parametric solution:

$$\boxed{\begin{aligned} x &= 1 - t \\ y &= 2 + 3t \\ z &= t \end{aligned}}$$

Definition: A matrix is in <u>row-echelon form</u> if it satisfies the following:

1.) All zero rows (if any) are at the bottom.

2.) The first nonzero entry from the left in each nonzero row is a 1 (called the leading 1).

3.) Each leading 1 is to the right of all leading ones above it.

A row-echelon matrix is in <u>reduced row-echelon form</u> if, in addition it satisfies:

4.) Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

row-echelon   reduced row-echelon

(How can you use elementary operations to get from RE to RRE?)

Theorem: Every matrix can be put in RE (and RRE) form via a sequence of elementary row operations, via the Gaussian Algorithm:

Gaussian Algorithm:

Step 1: If the matrix is 0 (i.e. all entries are 0), stop – it's already in RE (and RRE) form.

Step 2: Otherwise, find the first column from the left containing a nonzero entry, call it a. Move the row containing that entry to the top row.

Step 3: Multiply top row by $1/a$ to create a leading 1.

Step 4: Subtract multiples of that row from rows below it to make all entries below the leading 1 zero.

Step 5: Repeat 1-4 on matrix consisting of remaining rows.

The process stops when either no rows remain at step 5 or the matrix is 0.

Ex: Let's use the Gaussian algorithm on: $\left[\begin{array}{cccc|c} 0 & 0 & 2 & 4 & -1 \\ 0 & 3 & 0 & 9 & 0 \\ 0 & 2 & 1 & 5 & -2 \end{array}\right]$

Step 1: N/A

Step 2:
$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 4 & -1 \\ 0 & 3 & 0 & 9 & 0 \\ 0 & 2 & 1 & 5 & -2 \end{array}\right] \xrightarrow[\text{1 and 2}]{\text{swap}} \left[\begin{array}{cccc|c} 0 & 3 & 0 & 9 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 2 & 1 & 5 & -2 \end{array}\right]$$

first column containing a nonzero entry (let a = 3)

Step 3: Multiply the top row by $\frac{1}{3}$:

$$\xrightarrow{\frac{1}{3}1} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 2 & 1 & 5 & -2 \end{array}\right]$$

Step 4: Subtract 2 times row 1 from row 3:

$$\xrightarrow{3-2\,1} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right]$$

Step 5: Repeat for next two rows:

Step 1: N/A

Step 2:
$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ \hline 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right]$$

first nonzero column. Top entry is already $\neq 0$.
Set a = 2

Step 3: Multiply row 2 by ½

$$\xrightarrow{\frac{1}{2}(2)} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & -2 \end{array}\right]$$

Step 4:

$$\xrightarrow{(3)-(2)} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & -3 & -\frac{3}{2} \end{array}\right]$$

Step 5: Repeat on remaining row:

Step 2:

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & -3 & -\frac{3}{2} \end{array}\right]$$

first nonzero column $(a = -3)$

Step 3:

$$\xrightarrow{-\frac{1}{3}(3)} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Step 4,5: Done — no more rows.

In order to solve the system, we can continue by putting the matrix in RRE form:

$$\xrightarrow{①-3③} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 2 & -1/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right] \xrightarrow{②-2③} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 0 & -3/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right]$$

solution:

$$x_1 = t$$
$$x_2 = -3/2$$
$$x_3 = -3/2$$
$$x_4 = 1/2$$

x_1 can be anything!

What if we end up instead with the following matrix in RRE form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

converting this back to equations, we get:

$$x_1 = 0$$
$$x_3 = 0$$
$$0 = 1$$ obviously false.

Since there is <u>no</u> solution that makes this true, this system is inconsistent.

This general method for solving systems of equations is called Gaussian Elimination.

Gaussian elimination: To solve a system of equations...

1. Use elementary row operations to get a RRE matrix.

2. If a row $[0\ 0\ \ldots\ 0\ 1]$ occurs, the system is inconsistent.

3. Otherwise assign non-leading variables as parameters and solve for the leading variables in terms of the parameters.

Ex:
$$\left[\begin{array}{ccc|c} 1 & 1 & 7 & 0 \\ 2 & 3 & 2 & 1 \end{array}\right] \xrightarrow{②-2①} \left[\begin{array}{ccc|c} 1 & 1 & 7 & 0 \\ 0 & 1 & -12 & 1 \end{array}\right] \text{ (RE form)}$$

$$\xrightarrow{①-②} \left[\begin{array}{ccc|c} 1 & 0 & 19 & -1 \\ 0 & 1 & -12 & 1 \end{array}\right]$$

leading variables: x, y. Non-leading variable: z.

$$x + 19z = -1 \implies x = -1 - 19z$$
$$y - 12z = 1 \implies y = 1 + 12z$$

Set $z = t$.

Solution:

$$\boxed{\begin{aligned} x &= -1 - 19t \\ y &= 1 + 12t \\ z &= t \end{aligned}}$$

Rank

The rank of a matrix is the number of leading ones when it is put in row echelon form.

Ex: What's the rank of $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 4 & 6 & 4 \end{bmatrix}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 4 & 6 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 6 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 2

Note that if a matrix has m rows and n columns, the rank $r \leq m$ since the leading 1s are in different rows, and $r \leq n$ since they are in different columns.

Rank of augmented matrices

Suppose we have a consistent system of m equations in n variables. Let r = rank of corresponding augmented matrix.

$$m\left\{\underbrace{\begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{n}\right.\left|\begin{matrix} * \\ * \\ * \end{matrix}\right]$$

r = # of leading ones, n = total # of variables, so

n − r = # of nonleading variables = # of parameters in solution

$\Rightarrow$ If n = r, there are no parameters, and thus a unique solution.

If r < n, there is at least one parameter involved, which can take on infinitely many values, so there are infinitely many solutions.

Practice problems: 1.2.2 b, 1.2.3 b, 1.2.5 gh, 1.2.8, 1.2.11adf